UNIVERSITIES OF MANCHESTER LIVERPOOL LEEDS SHEFFIELD AND BIRMINGHAM

Joint Matriculation Board

General Certificate of Education

MATHEMATICS

SCHOLARSHIP

FRIDAY 28 JUNE 1957, 9.30-12.30

Answer **seven** *questions.*

Candidates need not confine their attention to the questions which correspond to the Alternative they offer in Mathematics, Advanced, Paper II. For the information of candidates these questions have A, B or C respectively in front of the number.

✓ **1.** (*a*) The first term of a geometric progression is a and the nth term is b. Obtain the common ratio, and express the product of the first n terms as simply as possible in terms of a, b and n.

(*b*) If s_n denotes the sum of the first n terms of the geometric series $x+x^2+x^3+\ldots$, show that

$$(1-x)\frac{ds_n}{dx} = (n+1)s_{n-1} - ns_n + 1.$$

✓ **2.** (*a*) Given that α and β are the roots of the equation

$$x^2-px+q=0,$$

form the equation whose roots are $(\alpha+\beta)^2$ and $-(\alpha-\beta)^2$.

(*b*) Given that x and y have opposite signs, solve the simultaneous equations

$$x+y+xy=-5,$$
$$x^2+y^2+x^2y^2=49.$$

✓ **3.** Suppose that this page is folded along a straight line so as to bring the top left hand corner on to the right hand edge of the page and so that the crease intersects the top edge of the page. If the length of the crease is x times the width of the page and the acute angle between the crease and the left hand edge is θ, show that

$$x=\frac{1}{2\cos^2\theta\sin\theta}.$$

Determine the minimum value of x as θ varies.

✓ **4.** A telescope which is initially horizontal and pointing south is rotated about a fixed axis until it is pointing south-east at an elevation of 30°. If the lens is at a distance of 5 ft. from the axis, find (i) the distance between the initial and final positions of the lens, and (ii) the length of the path followed by the lens, quoting three significant figures in each case.

✓ **5.** The points P_1, P_2 and P_3 lie on the rectangular hyperbola $x=ct$, $y=c/t$, and have the parameters t_1, t_2 and t_3 respectively. Show that the line through P_1 perpendicular to P_2P_3 has the equation

$$t_1t_2t_3\,x-t_1y+c(1-t_1{}^2t_2t_3)=0.$$

Deduce that the orthocentre H of the triangle $P_1P_2P_3$ lies on the hyperbola. Show also that, in the special case when P_2 and P_3 are the vertices of the hyperbola, the lines joining the origin to P_1 and H are equally inclined to the x and y axes respectively.

A6. Two small fixed pegs A and B are at a distance l apart, and the line AB is inclined at an angle α to the horizontal with B higher than A. A uniform rod of weight W is placed over A and under B, with its centre of gravity at a distance x from A, and it is kept in position by friction at the pegs, the coefficient of friction being $\frac{1}{5}$ at each peg. Find the components perpendicular to the rod of the forces exerted by the pegs, and show that, if $\tan \alpha > \frac{1}{5}$, the length of the rod must be at least

$$(1+5 \tan \alpha)l.$$

Determine the corresponding minimum length if $\tan \alpha < \frac{1}{5}$.

A7. A small artificial satellite is required to encircle the earth in the plane of the equator with uniform speed, the period of revolution being $2\pi/\omega$. The earth is assumed to be a sphere of radius R and the gravitational acceleration at height y above the earth's surface is $gR^2/(R+y)^2$.

(i) Show that, if $\omega^2 = g/(n^3R)$, the height of the satellite must be $(n-1)R$. If, in a particular case, this height is to be 300 miles, find to the nearest minute the period of revolution of the satellite, assuming that R is 3960 miles and g is 32 ft./sec.^{2}

(ii) Find the velocity, in terms of g, n and R, with which the satellite should be projected vertically from the earth's surface in order just to reach a height $(n-1)R$.

A8. A smooth light circular cylinder of radius a can rotate freely about its axis which is fixed in a horizontal position. A particle P of mass Nm is fixed to the outer surface of the cylinder, and a light inextensible string fastened to P passes over the cylinder and carries at its other end a particle of mass m hanging freely. Show that, if N is large, the system can be in equilibrium with the radius to P making a small angle $1/N$ radians approximately with the downward vertical.

Show that if the cylinder is rotated through a small angle from this position and then released from rest, it will subsequently perform simple harmonic motion, and find the frequency.

√ **A9.** A particle is projected horizontally with velocity V from the lowest point on the inner surface of a fixed smooth spherical shell of internal radius a, and it leaves the surface when its angle of elevation from the centre O of the shell is θ. Show that

$$V^2 = ga(2+3\sin\theta).$$

If the particle subsequently passes through O, find the value of $\sin\theta$.

A10. A straight narrow tube, of mass $2m$ and length l, has the shape of a right circular cylinder and is closed by plane ends A and B. A ball of mass m just fits inside the tube and can run smoothly along it. When the ball reaches an end it rebounds, the coefficient of restitution being e. Initially the tube is at rest on its side on a smooth horizontal floor, with the ball at rest at B. The tube is then suddenly set in motion in the direction AB with velocity V. Find the velocity of the tube when the ball is next leaving the end B, and find how far relative to the floor the ball has then travelled.

B11. (*a*) A variate x can take the values $1, 2, 3, \ldots, n$ with equal probability. Calculate the standard deviation of x.

(*b*) The standard deviations of two samples, each of n observations, are s_1 and s_2, their respective means being m_1 and m_2. Show that the standard deviation s of the combined sample of $2n$ observations is given by

$$s^2 = \tfrac{1}{2}(s_1^2+s_2^2)+\tfrac{1}{4}(m_1-m_2)^2.$$

B12. Four players A, B, C, D in this order throw a die in turn. Find for each player the probability of his being the first to throw a six. Explain how the sum of the four probabilities forms a useful check.

B13. The length x of the edge of a cube is rectangularly distributed between 5 and 10. Show that the volume y of the cube is distributed between 125 and 1,000 with a probability distribution $p(y)dy = \frac{1}{15}y^{-\frac{2}{3}}dy$.

Sketch the frequency curve, and calculate the mean and variance of the volume of the cube.

B14. A certain firm mass-produced machines which in the course of their assembly passed through four workshops A, B, C and D. A record of the times taken in each workshop and the times of transit from one workshop to the next was kept and from it the following summary showing the means and standard deviations of the times was published:

	Mean time (*hours*)	*Standard deviation* (*hours*)
Workshop A ..	3·48	0·25
Transit from A to B ..	0·23	0·05
Workshop B ..	4·56	0·30
Transit from B to C ..	0·53	0·12
Workshop C ..	1·91	0·20
Transit from C to D ..	0·32	0·10
Workshop D ..	2·67	0·20

Assuming that the workshop and transit times were independently normally distributed, calculate the mean and standard deviation of the times taken for the complete assembly of the machines, and show that only 1 per cent. of the machines were assembled in less than $12\frac{1}{2}$ hours while 6 per cent. took over $14\frac{1}{2}$ hours.

B15. If a sample of n_1 observations drawn from one population has a standard deviation s_1 and a sample of n_2 observations drawn from a second population has a standard deviation s_2, show that an estimate of the variance of the difference between the means of the two samples is

$$s_1^2/n_1+s_2^2/n_2,$$

provided that n_1 and n_2 are large.

Sixty boys who entered a school *A* and sixty boys who entered another school *B* were given the same examination in English. After each group of boys had attended their respective schools for one year they were each given another common examination in English. The means and standard deviations of the marks are shown in the following table:

	Examination mark of the 60 boys in each group			
	Upon entry		*After one year in the school*	
	Mean	*Standard deviation*	*Mean*	*Standard deviation*
School *A*	53	10	51	7
School *B*	50	10	48	7

Show that the difference between the means was not significant at the 5 per cent. level when the boys entered the schools but that it was significant at the 5 per cent. level after one year.

Interpret this result.

C16. (*a*) Prove that

$$\begin{vmatrix} (n-2)! & (n-1)! & n! \\ (n-1)! & n! & (n+1)! \\ n! & (n+1)! & (n+2)! \end{vmatrix} = 2(n-2)!(n-1)!n!$$

(*b*) Find the two values of t for which the equations

$$x-y+1 = 0,$$
$$x+ty+t = 0,$$
$$x-t^2y+t^2 = 0$$

can be simultaneously satisfied by finite values of x and y.

C17. (*a*) Find what restrictions must be imposed on the values of x and y in order to satisfy both the inequalities

$$x > y, \qquad \frac{x}{x+1} > \frac{y}{y+1}.$$

(*b*) Show that $b^2+c^2 \geqslant \frac{1}{2}(b+c)^2$, and hence find the range of values of a for which the simultaneous equations

$$a+b+c = 1,$$
$$a^2+b^2+c^2 = 3$$

may be satisfied by real values of b and c.

C18. (*a*) A number abc in the scale of 7 becomes cba when written in the scale of 9. Show that

$$b = 24a-40c,$$

and find the number in the scale of 10.

(*b*) If $n-1$ and $n+1$ are prime numbers greater than 5, show that $n(n^2-4)$ is divisible by 240.

C19. The lines OA, OB and OC are at right angles to one another and their lengths are a, b and c respectively. Show that the area of the triangle ABC is $\frac{1}{2}(b^2c^2+c^2a^2+a^2b^2)^{\frac{1}{2}}$. By considering the volume of the tetrahedron $OABC$, or otherwise, find the length of the perpendicular OP drawn from O to the plane ABC.

Show that CP is perpendicular to AB.

C20. Prove that the normal at a point P on an ellipse E whose foci are S and S' bisects the angle SPS'. Deduce that the normal meets the circle SPS' at a point on the minor axis.

If SP produced beyond P meets a line in T and PS' produced beyond S' meets the line in T', show that the ellipse through P whose foci are T and T' is orthogonal to E.

C21. (*a*) Show that, if $f(xt) = \frac{1}{x}f(t)$,

$$\int_x^{sx} f(t)dt = \int_1^s f(t)dt.$$

Hence, defining $\log x$ as $\int_1^x \frac{dt}{t}$, where $x > 0$, show that $\log xy = \log x + \log y$.

(*b*) Find $\int_1^2 (\log x)^2 dx$.

C22. (*a*) The normal at any point of a curve passes through the point (1, 1). Express this condition in the form of a differential equation, and hence find the equation of the family of curves which satisfy the condition.

(*b*) Given that y satisfies the differential equation

$$\frac{dy}{dx} + 2y \tan x = \sin x,$$

and that $y = 1$ when $x = \pi/3$, express dy/dx in terms of x.