UNIVERSITIES OF MANCHESTER LIVERPOOL
LEEDS SHEFFIELD AND BIRMINGHAM

Joint Matriculation Board

General Certificate of Education

MATHEMATICS. PAPER II

ADVANCED

ALTERNATIVE A. (Pages 2-4)

ALTERNATIVE B. (Pages 5-8)

ALTERNATIVE C. (Pages 9-12)

MONDAY 24 JUNE 1957, 2-5

Candidates must confine their attention to **one** *only of the Alternatives, and that the Alternative for which they have been entered on the entry-form.*

ALTERNATIVE A

Answer **seven** *questions.*

√ **A1.** Two uniform rods AB, BC, each of weight W and length $2a$, are smoothly hinged at B and rest in one horizontal line on supports at P and Q, where $PB = x$ and $BQ = y$. Prove that

$$\frac{1}{x} + \frac{1}{y} = \frac{2}{a},$$

and that $y \geqslant 2a/3$.

If the support at P cannot withstand a load greater than $\frac{3}{4}W$, prove that $y \leqslant 4a/5$.

A2. A uniform circular cylinder of weight W, whose axis is horizontal, rests on a fixed plane inclined to the horizontal at an angle α. A uniform rod, also of weight W, rests in a horizontal position with one end on the highest generator of the cylinder and the other smoothly hinged to the plane. The rod lies in a vertical plane through the centre of gravity of the cylinder and perpendicular to its axis. The system is in equilibrium and the coefficient of friction between the rod and cylinder and also between the cylinder and inclined plane is μ. Show that the normal reaction of the rod on the cylinder has magnitude $\frac{1}{2}W$. Show also that the frictional forces acting on the cylinder are equal in magnitude.

Hence show that $\mu \geqslant 3 \tan \frac{1}{2}\alpha$.

√ **A3.** When a motor launch moves northwards at 20 knots a pennant on its mast-head points due east. On the return journey, when the speed is 20 knots southwards, the pennant points N10°12′E. Find the speed and direction of the wind, assuming them to be unchanged throughout.

✓ **A4.** State clearly the relation between the acceleration and the displacement in simple harmonic motion.

A man of mass M stands on a horizontal platform which performs a vertical simple harmonic motion of period T and amplitude a. Find the force he exerts on the platform when the latter is at a height x above its mean position; give your answer in terms of M, g, x and T. Deduce that he maintains contact with the platform provided that $T \geqslant T_0$, where $T_0 = 2\pi\sqrt{(a/g)}$.

If $T = nT_0$, where $n > 1$, show that the greatest and least forces exerted by the man on the platform are in the ratio $(n^2+1)/(n^2-1)$.

✓ **A5.** A light inextensible string passes over a smooth fixed peg and carries at one end a particle A of mass $5m$ and at the other a light smooth pulley; over the latter passes a second light inextensible string carrying particles of masses $2m$ and $3m$ at its ends. Find the acceleration of A when the particles move vertically under gravity.

Find by how much the mass of A must be reduced in order that A can remain at rest while the other two particles are in motion.

✓ **A6.** (a) A number n of uniform wooden planks, each of weight W, thickness t and width a, lie in a pile of height nt on horizontal ground. The planks are taken from the pile and placed one on top of another to form a vertical wall of height na. Find the work done against gravity.

(b) Neglecting frictional losses, find the horsepower of a pump which raises water from a depth of 6 ft. and delivers 27·5 cu. ft. of water per minute at a speed of 32 ft./sec.

(Assume that g is 32 ft./sec.^{2}, and that the mass of 1 cu. ft. of water is 62·5 lb.)

✓ **A7.** A car of mass M lb. moves from rest on a horizontal road against a constant resistance of R poundals. The pull exerted by the engine decreases uniformly with the distance from the starting point, being initially P poundals and falling to R poundals after the car has travelled a ft. Derive an expression for the accelerating force when the car has moved x ft. from rest ($x < a$), and show that, if t sec. is the time taken to describe the distance x ft.,

$$\left(\frac{dx}{dt}\right)^2 = k^2(2ax - x^2),$$

where $k^2 = (P-R)/Ma$.

Verify by differentiation that $x = a(1-\cos kt)$ is a solution of the above equation, and show that it satisfies the initial conditions.

✓ **A8.** A particle P, of mass m, is suspended by two equal light inextensible strings PA, PB, where A and B are fixed at the same level and each string is inclined at an angle α to the horizontal. Find the tension in either string.

If the string PB is suddenly cut so that P starts to move in a circular path, find the tension in the string PA when it is inclined at an angle θ to the horizontal.

If the tension in PA is suddenly halved when PB is cut, find the angle α.

ALTERNATIVE B

Answer **six** *questions.*

2 sheets of graph paper supplied. Additional sheets will be supplied on request but **all** *sheets issued must be placed within the answer-book and handed in to the Supervisor.*

Statistical tables are also provided.

B1. The following table gives a frequency distribution of the times, to the nearest second, of the thirty-nine races (heats and final) for the 'Thames Challenge Cup' at the Henley Royal Regatta, 1956.

Time					*Frequency*
min.	*sec.*		*min.*	*sec.*	
7	10	to	7	19	4
7	20	,,	7	29	8
7	30	,,	7	39	8
7	40	,,	7	49	9
7	50	,,	7	59	6
8	0	,,	8	9	0
8	10	,,	8	19	3
8	20	,,	8	29	1

Plot a cumulative frequency graph and use it to determine

(i) the 80th percentile,

(ii) the percentile rank of a time of 7 min. 45 sec.

Explain the meaning of your results.

Calculate the mean time and the standard deviation.

B2. (*a*) A box contains ten radio valves all apparently sound, although four of them are actually substandard. Find the chance that, if two of the valves are taken from the box together, they are both substandard.

(*b*) When three marksmen take part in a shooting contest their chances of hitting the target are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$. Calculate the chance that one, and only one, bullet will hit the target if all three men fire at it simultaneously.

B3. The chance of contracting a certain disease in an industry is 1 in 7. Show that the chance that eight or more out of any ten workmen will contract the disease is about 1 in 168,000.

If the workmen of the industry are medically examined in groups of ten and the number of cases of the disease in each group is recorded, estimate the mean of these numbers and the standard deviation correct to three significant figures.

B4. A variable x is distributed at random between the values 0 and 4 so that the equation of the frequency curve is

$$y = Ax^3(4-x)^2,$$

where A is a constant. Find the value of A such that the area under the frequency curve is unity. Determine the mean and standard deviation of the distribution.

B5. *Associated TeleVision Limited*

Charges in £ for fifteen-second advertisements

Charge (£y) for a fifteen-second advertisement	*Number (x) of homes viewing ATV programmes, in thousands*							
	100–	200–	300–	400–	500–	600–	700–	800–
0–	6							
50–	2	5	2					
100–	5	7	3	1	1			
150–	2	6	1	3	1	2		
200–	1	17	7	0	1	0		
250–				2	1	1		
300–					1	1		
350–					1	0		
400–						3	1	
450–								

The above table shows how the charges for advertising, £*y*, made by ATV were related to the number of homes viewing, *x*, during the autumn of 1956. Thus the number 17 in the table indicates that there were 17 cases in which the charge was between £200 and £250 when between 200,000 and 300,000 homes were viewing.

For each range of *x* calculate to the nearest integer the mean value of *y*. Plot your results in a scatter diagram and sketch in the line of regression *y* on *x*.

Estimate the mean charge made by ATV for a fifteen-second advertisement at a time when half-a-million homes were viewing.

B6. Three dice, each numbered in the usual way from one to six, are coloured white, red and blue respectively. After casting them a boy ' scores ' in the following way. To the white number he adds twice the red number and then subtracts the blue number. Thus a white three, a red four and a blue two would score

$$3+8-2 = 9.$$

Assuming that the boy casts the dice a large number of times calculate the mean and variance of the scores.

B7. In order to find out whether the average speed of motor vehicles leaving London is different from that of motor vehicles entering London, cars and motor cycles were timed over a stretch of the Portsmouth road. The following table shows the results of the investigation :

	Leaving London	*Entering London*
Number of vehicles timed . .	50	50
Mean time in sec. . .	17·04	18·38
Variance in sec.^{2} . .	8·846	9·106

Determine whether the difference between the means is significant (i) at the 5 per cent. level, (ii) at the 1 per cent. level. Comment on the meaning of your result.

ALTERNATIVE C

Answer **seven** *questions.*

C1. Prove that

$$ax^2+bx+c$$

is positive for all real values of x if $a>0$ and $b^2<4ac$.

Find the range of values of k for which

$$x^2+kx+3+k$$

is positive for all real values of x. Deduce the range of values of k for which

$$k(x^2+kx+3+k)$$

is positive for all real values of x.

C2. (*a*) Sketch the graphs of

$$(x-1)(x-2) \text{ and } \frac{1}{(x-1)(x-2)}.$$

(The coordinates of turning points are not required.)

(*b*) If $y=x^3+7x-10$, show by considering the signs of y and dy/dx that y is zero for one and only one real value of x. Calculate this value of x correct to one decimal place.

C3. (*a*) Express the value of the determinant

$$\begin{vmatrix} 1 & -\cos\theta & -\sin\theta \\ 1 & \cos\theta & \sin\theta \\ 1 & \sin\theta & \cos\theta \end{vmatrix}$$

in terms of $\cos 2\theta$.

(*b*) Solve the equation

$$\begin{vmatrix} 1 & 1 & 1 \\ x & 2 & 3 \\ x^3 & 8 & 27 \end{vmatrix} = 0.$$

C10. Prove that

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \left\{ f(x)+f(-x) \right\} dx.$$

Hence show that

$$\int_{-\pi/4}^{\pi/4} \frac{1}{1+\sin x}\,dx = 2\int_{0}^{\pi/4} \sec^2 x\,dx,$$

and evaluate this integral.

Use the first result to evaluate

$$\int_{-1}^{1} \frac{1}{1+e^{-x}}\,dx.$$

C11. (*a*) Find the general solution of the differential equation

$$y(x-1)\frac{dy}{dx} = y^2+1.$$

(*b*) Given that y satisfies the differential equation

$$(t+1)\frac{dy}{dt}+y = t\sin t,$$

and that $y = 1$ when $t = 0$, find y in terms of t.